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A Modified Iterative Scheme for Common Fixed Points of δ-Enriched Asymptotically Quasi-I-Nonexpansive Mappings in Banach Spaces

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Abstract

This paper introduces a new implicit iterative process for approximating common fixed points of a δ-enriched asymptotically quasi-I-nonexpansive nonself mapping T and a δ-enriched asymptotically quasi-nonexpansive mapping I in real Banach spaces. Under appropriate conditions, we prove that $\lim_{n\to\infty} \|u_n - p\|$ exists. Strong convergence is characterized by $\liminf d(u_n, \mathcal{F}) = 0$. In uniformly convex Banach spaces, we establish that $\|u_n - Tu_n\| \to 0$ and $\|u_n - Iu_n\| \to 0$. Weak convergence is obtained under Opial's condition and demiclosedness, while strong convergence follows from semicompactness of either mapping. The optimal enrichment parameter was derived and the stability under perturbations was proved. The results herein unify and extend several classical convergence theorems in fixed point theory.

Keywords: Enriched Asymptotically Nonexpansive Mappings, Common Fixed Points, Enriched Contractions, Implicit Iteration Schemes, Uniformly Convex Banach Spaces.

1 Introduction

Let C be a nonempty subset of a real normed linear space E and let $T : C \to C$ be a mapping. Denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{u \in C : Tu = u\}$. Throughout this paper, we assume that $F(T) \neq \emptyset$. We begin by recalling some standard definitions.

Definition 1.1. A mapping $T : C \to C$ is said to be:

(i) *nonexpansive* if $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in C$;

(ii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n u - T^n v\| \leq k_n \|u - v\|$ for all $u, v \in C$ and $n \in \mathbb{N}$;

(iii) *quasi-nonexpansive* if $\|Tu - p\| \leq \|u - p\|$ for all $u \in C$ and $p \in F(T)$;

(iv) *asymptotically quasi-nonexpansive* if there exists a sequence $\{\eta_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \eta_n = 1$ such that $\|T^n u - p\| \leq \eta_n \|u - p\|$ for all $u \in C$, $p \in F(T)$, and $n \in \mathbb{N}$.

From the above definitions, it follows that every nonexpansive mapping is asymptotically nonexpansive, and every asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. However, the converse implications do not hold in general (see Berinde, 2019).

If C is a closed nonempty subset of a Banach space and $T : C \to C$ is nonexpansive, it is well known that T may fail to have a fixed point (unlike the case of strict contractions). Moreover, even when a fixed point exists, the Picard sequence $\{u_n\}$ defined by $u_{n+1} = Tu_n$ may not converge to it.

In the seminal works (Browder, 1965, 1967), Browder studied iterative constructions for fixed points of nonexpansive mappings defined on closed convex subsets of Hilbert spaces. Over the past three decades, the study of iterative approximation methods for fixed points of nonexpansive mappings and their generalizations has flourished, attracting the attention of many researchers (see, e.g., Chidume, 2009; Berinde, 2019 for comprehensive overviews).

Diaz and Metcalf (1967) investigated quasi-nonexpansive mappings in Banach spaces. Ghosh and Debnath (1997) established necessary and sufficient conditions for the convergence of Ishikawa iterates for quasi-nonexpansive mappings on closed convex subsets of Banach spaces. Iterative approximation problems for nonexpansive, asymptotically nonexpansive, and asymptotically quasi-nonexpansive mappings have been extensively studied by Goebel and Kirk (1972), Liu (2001), Wittmann (1992), Reich (1980), Gornicki (1989), Schu (1991), Shioji and Takahashi (1998), and Tan and Xu (1993) in the settings of Hilbert spaces and uniformly convex Banach spaces.

Various methods exist for approximating fixed points of nonexpansive mappings. Xu and Ori (2001) introduced an implicit iteration process for approximating a common fixed point of a finite family of nonexpansive mappings in Hilbert spaces. Subsequently, Sun (2003) extended this implicit iteration process to the case of asymptotically quasi-nonexpansive mappings in Banach spaces.

In Gu and Lu (2006), weak and strong convergence of an implicit iteration process with errors was studied for a finite family of nonexpansive mappings in Banach spaces, extending and improving several earlier results (see also Li & Li, 2009; Zhang & Su, 2007 for applications and other implicit iteration methods).

Several concepts generalize the notion of nonexpansive mappings. One such concept is I-nonexpansivity, introduced in Shahzad (2004). We recall the relevant definitions.

Definition 1.2. Let $T : C \to C$ and $I : C \to C$ be two mappings defined on a nonempty subset C of a real normed linear space E. Then T is said to be:

(i) *I-nonexpansive* if $\|Tu - Tv\| \leq \|Iu - Iv\|$ for all $u, v \in C$;

(ii) *asymptotically I-nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n u - T^n v\| \leq k_n \|I^n u - I^n v\|$ for all $u, v \in C$ and $n \geq 1$;

(iii) *asymptotically quasi I-nonexpansive* if there exists a sequence $\{\eta_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \eta_n = 1$ such that $\|T^n u - p\| \leq \eta_n \|I^n u - p\|$ for all $u \in C$, $p \in F(T) \cap F(I)$, and $n \geq 1$.

Remark 1.3. If $F(T) \cap F(I) \neq \emptyset$, then every asymptotically I-nonexpansive mapping is asymptotically quasi-I-nonexpansive. However, there exist nonlinear continuous asymptotically quasi-I-nonexpansive mappings that are not asymptotically I-nonexpansive.

In Temir and Gul (2007), a weak convergence theorem for I-asymptotically quasi-nonexpansive mappings in Hilbert spaces was proved. Strong convergence of Mann iterations for I-nonexpansive mappings was established in Rhodes and Temir (2006). Best approximation properties of I-nonexpansive mappings were investigated in Shahzad (2004). In Kumam, Kumethong, and Jewwaiworn (2008), weak convergence of a three-step Noor iterative scheme for an I-nonexpansive mapping in Banach spaces was demonstrated. Recently, Temir (2009) studied weak and strong convergence of an implicit iteration process for a common fixed point of a finite family of I-asymptotically nonexpansive mappings. For a single I-asymptotically nonexpansive mapping T, the iteration method considered in Temir (2009) is defined by:

$$\begin{cases} u_1 \in C, \\ u_{n+1} = (1-\alpha_n)u_n + \alpha_n I^n v_n, \\ v_n = (1-\beta_n)u_n + \beta_n T^n u_n, \quad n \geq 1, \end{cases} \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

In recent developments, Mukhamedov and Saburov (2010) extended the work of Sun (2003) to I-asymptotically quasi-nonexpansive mappings defined on uniformly convex Banach spaces. They studied weak and strong convergence of the following iteration scheme:

$$\begin{cases} u_0 \in C, \\ u_n = (1-\alpha_n)u_{n-1} + \alpha_n T^n v_n, \\ v_n = (1-\beta_n)u_n + \beta_n I^n u_n, \quad n \geq 1. \end{cases} \tag{2}$$

Definition 1.4. A subset C of a Banach space X is a **retract** of X if there exists a continuous mapping $R : X \to C$ (a **retraction**) such that $R(u) = u$ for all $u \in C$. If R is nonexpansive, it is called a **nonexpansive retraction**.

If $R : X \to C$ is a retraction, then $R^2 = R$. Every closed convex subset of a uniformly convex Banach space is a retract.

Definition 1.5. A mapping $I : C \to C$ is called a **δ-enriched asymptotically nonexpansive mapping** if there exists $\delta \in [0, \infty)$ and a sequence $\{\eta_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \eta_n = 1$ such that

$$\|\delta(u-v) + I^n(u) - I^n(v)\| \leq (\delta+1)\eta_n \|u-v\|, \quad \forall u, v \in C, \ \forall n \in \mathbb{N}.$$

Setting $\delta = \frac{1}{d} - 1$ for $d \in (0,1]$, i.e., $d = \frac{1}{\delta+1}$, we obtain

$$\|(1-d)(u-v) + dI^n(u) - dI^n(v)\| \leq \eta_n \|u-v\|,$$

which is equivalent to

$$\|I_d^n(u) - I_d^n(v)\| \leq \eta_n \|u - v\|,$$

where $I_d^n = (1-d)I^* + dI^n$ and I^* is also an identity mapping.

Definition 1.6. A mapping $T : C \to E$ with retract R is called a **δ-enriched asymptotically I-nonexpansive mapping** if there exists $\delta \in [0, \infty)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|\delta(u-v)+T(RT)^{n-1}(u)-T(RT)^{n-1}(v)\| \leq (\delta+1)k_n\|\delta(u-v)+I^n u-I^n v\|, \quad \forall u, v \in C,\ \forall n \in \mathbb{N}. \tag{3}$$

Setting $\delta = \frac{1}{d} - 1$ for $d \in (0, 1]$, i.e., $d = \frac{1}{\delta+1}$, we obtain

$$\|(1-d)(u-v) + dT(RT)^{n-1}(u) - dT(RT)^{n-1}(v)\| \leq \frac{k_n}{d}\|(1-d)(u-v) + dI^n u - dIv\|,$$

which is equivalent to

$$\|T_d(RT_d)^{n-1}(u) - T_d(RT_d)^{n-1}(v)\| \leq \frac{k_n}{d}\|I_d^n u - I_d^n v\|, \tag{4}$$

where $T_d(RT_d)^{n-1} = (1-d)I^* + dT(RT)^{n-1}$, $I_d^n = (1-d)I^* + dI^n$ and I^* is also an identity mapping.

Definition 1.7. A mapping $I : C \to C$ is called a **δ-enriched asymptotically quasi-nonexpansive mapping** if there exists $\delta \in [0, \infty)$ and a sequence $\{\eta_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} \eta_n = 1$ such that

$$\|\delta(u-p) + I^n(u) - p\| \leq (\delta+1)\eta_n\|u-p\|, \quad \forall u \in C, p \in F(I),\ n \in \mathbb{N}.$$

Setting $\delta = \frac{1}{d} - 1$ for $d \in (0, 1]$, i.e., $d = \frac{1}{\delta+1}$, we obtain

$$\|(1-d)(u-p) + dI^n(u) - p\| \leq \eta_n\|u-p\|,$$

which is equivalent to

$$\|I_d^n(u) - p\| \leq \eta_n\|u-p\|,$$

where $I_d^n = (1-d)I^* + dI^n$ and I^* is also an identity mapping.

Definition 1.8. A mapping $T : C \to E$ with retract R is called a *δ-enriched asymptotically quasi I-nonexpansive nonself mapping* if there exists $\delta \in [0, \infty)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|\delta(u-p) + T(RT)^{n-1}u - p\| \leq (\delta+1)k_n\|\delta(u-p) + I^n u - p\| \tag{5}$$

for all $u \in C$, $p \in F(T) \cap F(I)$, and $n \geq 1$.

Similarly, setting $\delta = \frac{1}{d} - 1$ with $d \in (0, 1]$, i.e., $d = \frac{1}{\delta+1}$, we obtain

$$\|(1-d)(u-p) + dT(RT)^{n-1}(u) - p\| \leq \frac{k_n}{d}\|(1-d)(u-p) + dI^n u - p\|, \tag{6}$$

which gives

$$\|T_d(RT_d)^{n-1}(u) - p\| \leq \frac{k_n}{d}\|I_d^n u - p\|, \tag{7}$$

where $T_d(RT_d)^{n-1} = (1-d)I^* + dT(RT)^{n-1}$, $I_d^n = (1-d)I^* + dI^n$ and I^* is also an identity mapping.

In this paper, we extend the implicit iterative process introduced in Sun (2003) and Mukhamedov and Saburov (2010) to δ-enriched asymptotically quasi I-nonexpansive mappings in uniformly convex Banach spaces. Specifically, let C be a nonempty convex subset of a real Banach space E, let $T : C \to E$ be a δ-enriched asymptotically quasi I-nonexpansive nonself mapping, and let $I : C \to C$ be a δ-enriched asymptotically quasi-nonexpansive mapping. For given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$, we consider the following iteration scheme:

$$\begin{cases} u_0 \in C, \\ u_n = (1-\alpha_n)u_{n-1} + \alpha_n T_d(RT_d)^{n-1} v_n, \\ v_n = (1-\beta_n)u_n + \beta_n I_d^n u_n, \quad n \geq 1, \end{cases} \tag{8}$$

where $T_d(RT_d)^{n-1} = (1-d)I + dT(RT)^{n-1}$, $I_d^n = (1-d)I + dI^n$, and $d \in (0, 1]$ is given by $d = \frac{1}{\delta+1}$ for $\delta \in [0, \infty)$.

We prove weak and strong convergence of the implicit iterative process (8) to a common fixed point of T and I. Our results generalize and extend the corresponding main results of Xu and Ori (2001), Sun (2003), Gu and Lu (2006), and Mukhamedov and Saburov (2010).

2 Theory and Methods

Throughout this paper, we assume that E is a real Banach space. The set of fixed points and the domain of a mapping T are denoted by $F(T)$ and $D(T)$, respectively. A Banach space E is said to satisfy Opial's condition (Opial, 1967) if, for every sequence $\{u_n\}$ in E converging weakly to u, the inequality

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\| \tag{9}$$

holds for all $v \in E$ with $v \neq u$. It is well known (see Lami Dozo, 1973) that inequality (9) is equivalent to

$$\limsup_{n\to\infty} \|u_n - u\| < \limsup_{n\to\infty} \|u_n - v\|. \tag{10}$$

Definition 2.1. Let C be a closed subset of a real Banach space E and let $T : C \to C$ be a mapping.

(i) T is said to be *semiclosed* (or *demiclosed*) at zero if, for any bounded sequence $\{u_n\}$ in C

such that $u_n \rightharpoonup u \in C$ and $Tu_n \to 0$ strongly, we have $Tu = 0$.

(ii) T is called *semicompact* if, for any bounded sequence $\{u_n\}$ in C satisfying $\|u_n - Tu_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ that converges strongly to some $u^* \in C$.

(iii) T is said to be a *uniformly L-Lipschitzian mapping* if there exists a constant $L > 0$ such that

$$\|T^n u - T^n v\| \leq L\|u - v\|$$

for all $u, v \in C$ and all integers $n \geq 1$.

The following lemmas play a crucial role in proving our main results.

Lemma 2.2 (see Schu, 1991)**.** *Let E be a uniformly convex Banach space, and let b, c be constants with $0 < b < c < 1$. Suppose $\{t_n\}$ is a sequence in (b, c) and $\{u_n\}, \{v_n\}$ are sequences in E such that*

$$\lim_{n\to\infty} \|t_n u_n + (1 - t_n)v_n\| = d, \quad \limsup_{n\to\infty} \|u_n\| \leq d, \quad \limsup_{n\to\infty} \|v_n\| \leq d \tag{11}$$

for some $d \geq 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Lemma 2.3 (see Tan & Xu, 1993)**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{\rho_n\}$ be three sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \rho_n < \infty$. If either of the following conditions holds:*

(i) $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$;

(ii) $a_{n+1} \leq (1 + b_n)a_n + \rho_n$ for all $n \geq 1$;

then the limit $\lim_{n\to\infty} a_n$ exists. If some subsequence of $\{a_n\}$ converges to 0, then $\lim_{n\to\infty} a_n = 0$.

3 Main Results and Discussion

In this section we will prove our main results. To formulate them, we need some auxiliary results.

Lemma 3.1. *Let E be a real Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to E$ with retraction R be a δ–enriched asymptotically quasi I-nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $I : C \to C$ be a δ–enriched asymptotically quasi-nonexpansive mapping with a sequence $\{\eta_n\} \subset [1, \infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $\sigma^* = \sup_n \alpha_n$, $\zeta = \sup_n k_n \geq 1$, $B = \sup_n \eta_n \geq 1$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ and $d \in (0, 1]$ which satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty} (\frac{k_n}{d}\eta_n - 1)\alpha_n < \infty$,

(ii) $\sigma^* < \frac{1}{\frac{\zeta^2}{d^2}B^2}$.

If $\{u_n\}$ *is the implicit iterative sequence defined by* (8)*, then for each* $p \in \mathcal{F} = F(T) \cap F(I)$ *the limit* $\lim_{n\to\infty} \|u_n - p\|$ *exists.*

Proof. Since $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$, for any given $p \in \mathcal{F}$, it follows from (8) that

$$\begin{aligned}
\|u_n - p\| &= \Big\|(1-\alpha_n)(u_{n-1} - p) + \alpha_n(T_d(RT_d)^{n-1}v_n - p)\Big\| \\
&\leq (1-\alpha_n)\|u_{n-1} - p\| + \alpha_n\|T_d(RT_d)^{n-1}v_n - p\| \\
&\leq (1-\alpha_n)\|u_{n-1} - p\| + \alpha_n\frac{k_n}{d}\|I_d^n v_n - p\| \\
&= (1-\alpha_n)\|u_{n-1} - p\| + \alpha_n\frac{k_n}{d}\|(1-d)v_n + dI^n v_n - p\| \\
&= (1-\alpha_n)\|u_{n-1} - p\| + \alpha_n\frac{k_n}{d}\Big\|(1-\frac{1}{\delta+1})v_n + \frac{1}{\delta+1}I^n v_n - p\Big\| \\
&= (1-\alpha_n)\|u_{n-1} - p\| + \alpha_n\frac{k_n}{d}\frac{1}{\delta+1}\|\delta(v_n - p) + I^n v_n - p\| \\
&\leq (1-\alpha_n)\|u_{n-1} - p\| + \alpha_n\frac{k_n}{d}\eta_n\|v_n - p\|. \qquad (12)
\end{aligned}$$

Again from (8) we observe that

$$\begin{aligned}
\|v_n - p\| &= \|(1-\beta_n)(u_n - p) + \beta_n(I_d^n u_n - p)\| \\
&\leq (1-\beta_n)\|u_n - p\| + \beta_n\|I_d^n u_n - p\| \\
&= (1-\beta_n)\|u_n - p\| + \beta_n\|(1-d)u_n + dI^n u_n - p\| \\
&= (1-\beta_n)\|u_n - p\| + \beta_n\Big\|(1-\frac{1}{\delta+1})u_n + \frac{1}{\delta+1}I^n u_n - p\Big\| \\
&= (1-\beta_n)\|u_n - p\| + \beta_n\frac{1}{\delta+1}\|\delta(u_n - p) + I^n u_n - p\| \\
&\leq (1-\beta_n)\|u_n - p\| + \beta_n\eta_n\|u_n - p\| \\
&\leq (1-\beta_n)\eta_n\|u_n - p\| + \beta_n\eta_n\|u_n - p\| \\
&\leq \eta_n\|u_n - p\|, \qquad (13)
\end{aligned}$$

which implies that

$$\begin{aligned}
\|v_n - p\| &\leq \eta_n\|u_n - p\| \\
&\leq \frac{k_n}{d}\eta_n\|u_n - p\|. \qquad (14)
\end{aligned}$$

Now, putting (14) into (12), we obtain

$$\|u_n - p\| \leq (1-\alpha_n)\|u_{n-1} - p\| + \alpha_n\frac{k_n^2}{d^2}\eta_n^2\|u_n - p\|, \qquad (15)$$

and then, we have

$$\left(1-\alpha_n\frac{k_n^2}{d^2}\eta_n^2\right)\|u_n-p\| \leq (1-\alpha_n)\|u_{n-1}-p\|. \tag{16}$$

From condition (ii) above, we get $1 > \sigma^*\frac{\zeta^2}{d^2}B^2 \geq \alpha_n\frac{k_n^2}{d^2}\eta_n^2$, and thus

$$1-\alpha_n\frac{k_n^2}{d^2}\eta_n^2 \geq 1-\sigma^*\frac{\zeta^2}{d^2}B^2 > 0. \tag{17}$$

Furthermore, from (16) and (17) we obtain

$$\begin{aligned}
\|u_n-p\| &\leq \frac{1-\alpha_n}{1-\alpha_n\frac{k_n^2}{d^2}\eta_n^2}\|u_{n-1}-p\| \\
&= \frac{1-\alpha_n\frac{k_n^2}{d^2}\eta_n^2+\alpha_n\frac{k_n^2}{d^2}\eta_n^2-\alpha_n}{1-\alpha_n\frac{k_n^2}{d^2}\eta_n^2}\|u_{n-1}-p\| \\
&\leq \left[1+\frac{(\frac{k_n^2}{d^2}\eta_n^2-1)\alpha_n}{1-\alpha_n\frac{k_n^2}{d^2}\eta_n^2}\right]\|u_{n-1}-p\| \\
&\leq \left[1+\frac{(\frac{k_n^2}{d^2}\eta_n^2-1)\alpha_n}{1-\sigma^*\frac{\zeta^2}{d^2}B^2}\right]\|u_{n-1}-p\|.
\end{aligned} \tag{18}$$

Choosing $\phi_n = \frac{(\frac{k_n^2}{d^2}\eta_n^2-1)\alpha_n}{1-\sigma^*\frac{\zeta^2}{d^2}B^2}$, we observe that (18) becomes

$$\|u_n-p\| \leq (1+\phi_n)\|u_{n-1}-p\|. \tag{19}$$

Hence, from condition (i) we observe that

$$\begin{aligned}
\sum_{n=1}^{\infty}\phi_n &= \frac{1}{1-\sigma^*\frac{\zeta^2}{d^2}B^2}\sum_{n=1}^{\infty}(\frac{k_n^2}{d^2}\eta_n^2-1)\alpha_n \\
&= \frac{1}{1-\sigma^*\frac{\zeta^2}{d^2}B^2}\sum_{n=1}^{\infty}(\frac{k_n}{d}\eta_n-1)(\frac{k_n}{d}\eta_n+1)\alpha_n \\
&\leq \frac{\frac{\zeta}{d}B+1}{1-\sigma^*\frac{\zeta^2}{d^2}B^2}\sum_{n=1}^{\infty}(\frac{k_n}{d}\eta_n-1)\alpha_n < \infty.
\end{aligned} \tag{20}$$

Now, let $x_n = \|u_{n-1}-p\|$ in (19) then we obtain

$$x_{n+1} \leq (1+\phi_n)x_n, \tag{21}$$

and applying Lemma 2.3 implies the existence of the limit $\lim_{n\to\infty} x_n$. Therefore,

$$\lim_{n\to\infty}\|u_n-p\| = r \tag{22}$$

exists, where $r \geq 0$ is a constant completing the proof of Lemma (3.1). □

With the above preparations, we now prove the following result.

Theorem 3.2. *Let E be a real Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to E$ with retraction R be a uniformly L_1-Lipschitzian δ–enriched asymptotically quasi-I-nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$ and let $I : C \to C$ be a uniformly L_2-Lipschitzian δ–enriched asymptotically quasi-nonexpansive mapping with a sequence $\{\eta_n\} \subset [1, \infty)$ such that*
$\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $\sigma^* = \sup_n \alpha_n$, $\zeta = \sup_n k_n \geq 1$, $B = \sup_n \eta_n \geq 1$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ and $d \in (0, 1]$ which satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} (\frac{k_n}{d}\eta_n - 1)\alpha_n < \infty$,

(ii) $\sigma^* < \frac{1}{\frac{\zeta^2}{d^2}B^2}$.

Then the implicit iterative sequence $\{u_n\}$ defined by (8) converges strongly to a common fixed point in $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$ if and only if

$$\liminf_{n\to\infty} d(u_n, \mathcal{F}) = 0. \tag{23}$$

Proof. While the necessity of condition (23) is evident, the sufficiency requires proof. We establish it as follows.

Since $T : C \to E, I : C \to C$ are uniformly Lipschitzian mappings, T and I are continuous mappings. Therefore the sets $F(T)$ and $F(I)$ are closed. Hence $\mathcal{F} = F(T) \cap F(I)$ is a nonempty closed set.

For any given $p \in \mathcal{F}$, from (19), we get

$$\|u_n - p\| \leq (1 + \phi_n)\|u_{n-1} - p\|, \tag{24}$$

here as before $\phi_n = \frac{(\frac{k_n^2}{d^2}\eta_n^2 - 1)\alpha_n}{1 - \sigma^* \frac{\zeta^2}{d^2}B^2}$ with $\sum_{n=1}^{\infty} \phi_n < \infty$. Hence,

$$d(u_n, \mathcal{F}) \leq (1 + \phi_n)d(u_{n-1}, \mathcal{F}). \tag{25}$$

From (25) and applying Lemma 2.3 we obtain the existence of the limit $\lim_{n\to\infty} d(u_n, \mathcal{F})$. From condition (23) again, we obtain

$$\lim_{n\to\infty} d(u_n, \mathcal{F}) = \liminf_{n\to\infty} d(u_n, \mathcal{F}) = 0. \tag{26}$$

Furthermore, since $1 + t \leq \exp(t)$ for all $t \geq 0$, and from (24), we get

$$\|u_n - p\| \leq \exp(\phi_n)\|u_{n-1} - p\|. \tag{27}$$

Hence, for any positive integers m, n, from (27) with $\sum_{n=1}^{\infty} \phi_n < \infty$ we find

$$\begin{aligned}\|u_{n+m} - p\| &\leq \exp(\phi_{n+m})\|u_{n+m-1} - p\| \\ &\leq \exp(\phi_{n+m} + \phi_{n+m-1})\|u_{n+m-2} - p\| \\ &\vdots \\ &\leq \exp\left(\sum_{i=n+1}^{n+m} \phi_i\right)\|u_n - p\| \\ &\leq \exp\left(\sum_{i=1}^{\infty} \phi_i\right)\|u_n - p\|, \end{aligned} \tag{28}$$

thus, we obtain

$$\|u_{n+m} - p\| \leq \Upsilon\|u_n - p\| \tag{29}$$

for all $p \in \mathcal{F}$, where $\Upsilon = \exp\left(\sum_{i=1}^{\infty} \phi_i\right) < \infty$.

Since $\lim_{n\to\infty} d(u_n, \mathcal{F}) = 0$, then for any given $\varepsilon > 0$, there exists a positive integer number n_0 such that

$$d(u_{n_0}, \mathcal{F}) < \frac{\varepsilon}{\Upsilon}. \tag{30}$$

Therefore there exists $p_1 \in \mathcal{F}$ such that

$$\|u_{n_0} - p_1\| < \frac{\varepsilon}{\Upsilon}. \tag{31}$$

Consequently, for all $n \geq n_0$ from (29) we obtain

$$\|u_n - p_1\| \leq \Upsilon\|u_{n_0} - p_1\| < \Upsilon \cdot \frac{\varepsilon}{\Upsilon} = \varepsilon, \tag{32}$$

which implies that the sequence $\{u_n\}$ converges strongly to a common fixed point p_1 in $\mathcal{F}$. This completes the proof of Theorem (3.2). □

We need one more auxiliary result.

Proposition 3.3. *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to E$ with retraction R be a uniformly L_1-Lipschitzian δ–enriched asymptotically quasi-I-nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$ and let $I : C \to C$ be a uniformly L_2-Lipschitzian δ–enriched asymptotically quasi-nonexpansive mapping with a sequence $\{\eta_n\} \subset [1, \infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $\sigma_* = \inf_n \alpha_n$, $\sigma^* = \sup_n \alpha_n$, $\zeta = \sup_n k_n \geq 1$, $B = \sup_n \eta_n \geq 1$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ and $d \in (0, 1]$ which satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty}(\frac{k_n}{d}\eta_n - 1)\alpha_n < \infty$,

(ii) $0 < \sigma_* \leq \sigma^* < \frac{1}{\frac{\zeta^2}{d^2}B^2}$,

(iii) $0 < \beta_* = \inf_n \beta_n \leq \sup_n \beta_n = \beta^* < 1$.

Then the implicit iterative sequence $\{u_n\}$ *defined by* (8) *satisfies the following:*

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0 = \lim_{n\to\infty} \|u_n - Iu_n\|. \tag{33}$$

Proof. From Lemma (3.1) for any $p \in \mathcal{F} = F(T) \cap F(I)$ we have $\lim_{n\to\infty} \|u_n - p\| = r$. It follows from (8) that

$$\|u_n - p\| = \Big\|(1-\alpha_n)(u_{n-1} - p) + \alpha_n(T_d(RT_d)^{n-1}v_n - p)\Big\| \to r, \quad n \to \infty. \tag{34}$$

From the definition of δ–enriched asymptotically quasi-I-nonexpansivity of T and δ–enriched asymptotically quasi-nonexpansivity of I and (14) we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|T_d(RT_d)^{n-1}v_n - p\| &\leq \limsup_{n\to\infty} \frac{k_n}{d}\eta_n\|v_n - p\| \\ &\leq \limsup_{n\to\infty} \frac{k_n^2}{d^2}\eta_n^2\|u_n - p\| = r.\end{aligned} \tag{35}$$

Now using

$$\limsup_{n\to\infty} \|u_{n-1} - p\| = r \tag{36}$$

with (35) and applying Lemma 2.2 to (34) we observe that

$$\lim_{n\to\infty} \|u_{n-1} - T_d(RT_d)^{n-1}v_n\| = 0. \tag{37}$$

Now from (8) and (37) we have that

$$\lim_{n\to\infty} \|u_n - u_{n-1}\| = \lim_{n\to\infty} \|\alpha_n(T_d(RT_d)^{n-1}v_n - u_{n-1})\| = 0. \tag{38}$$

On the other hand, we have

$$\begin{aligned}\|u_{n-1} - p\| &\leq \|u_{n-1} - T_d(RT_d)^{n-1}v_n\| + \|T_d(RT_d)^{n-1}v_n - p\| \\ &\leq \|u_{n-1} - T_d(RT_d)^{n-1}v_n\| + \frac{k_n}{d}\eta_n\|v_n - p\|,\end{aligned} \tag{39}$$

which implies

$$\|u_{n-1} - p\| - \|u_{n-1} - T_d(RT_d)^{n-1}v_n\| \leq \frac{k_n}{d}\eta_n\|v_n - p\|. \tag{40}$$

(40) with (14) implies that

$$\|u_{n-1}-p\|-\|u_{n-1}-T_d(RT_d)^{n-1}v_n\| \leq \frac{k_n}{d}\eta_n\|v_n-p\| \leq \frac{k_n^2}{d^2}\eta_n^2\|u_n-p\|. \tag{41}$$

Then (37) and (35) imply that

$$\lim_{n\to\infty}\|v_n-p\|=r. \tag{42}$$

Again from (8) we can see that

$$\|v_n-p\|=\|(1-\beta_n)(u_n-p)+\beta_n(I_d^n u_n-p)\|\to r, \quad n\to\infty. \tag{43}$$

From (22) one finds

$$\limsup_{n\to\infty}\|I_d^n u_n-p\|\leq\limsup_{n\to\infty}\eta_n\|u_n-p\|=r. \tag{44}$$

Now applying Lemma 2.2 to (43) we obtain

$$\lim_{n\to\infty}\|u_n-I_d^n u_n\|=0. \tag{45}$$

Consider

$$\begin{aligned}\|u_n-T_d(RT_d)^{n-1}u_n\| &\leq \|u_n-u_{n-1}\|+\|u_{n-1}-T_d(RT_d)^{n-1}v_n\|+\|T_d(RT_d)^{n-1}v_n-T_d(RT_d)^{n-1}u_n\|\\ &\leq \|u_n-u_{n-1}\|+\|u_{n-1}-T_d(RT_d)^{n-1}v_n\|+\frac{L_1}{d}\|v_n-u_n\|\\ &= \|u_n-u_{n-1}\|+\|u_{n-1}-T_d(RT_d)^{n-1}v_n\|+\frac{L_1}{d}\|\beta_n(I_d^n u_n-u_n)\|\\ &\leq \|u_n-u_{n-1}\|+\|u_{n-1}-T_d(RT_d)^{n-1}v_n\|+\frac{L_1}{d}\beta_n\|I_d^n u_n-u_n\|. \end{aligned}\tag{46}$$

Then from (37), (38), and (45) we get

$$\lim_{n\to\infty}\|u_n-T_d(RT_d)^{n-1}u_n\|=0. \tag{47}$$

Finally, from

$$\begin{aligned}\|u_n - T_d u_n\| &\leq \|u_n - T_d(RT_d)^{n-1}u_n\| + \|T_d(RT_d)^{n-1}u_n - T_d u_n\| \\ &\leq \|u_n - T_d(RT_d)^{n-1}u_n\| + \frac{L_1}{d}\|(RT_d)^{n-1}u_n - u_n\| \\ &\leq \|u_n - T_d(RT_d)^{n-1}u_n\| + \frac{L_1}{d}(\|(RT_d)^{n-1}u_n - (RT_d)^{n-1}u_{n-1}\| \\ &\quad + \|(RT_d)^{n-1}u_{n-1} - u_{n-1}\| + \|u_{n-1} - u_n\|) \\ &\leq \|u_n - T_d(RT_d)^{n-1}u_n\| + \frac{L_1}{d}(\frac{L_1}{d}\|u_n - u_{n-1}\| \\ &\quad + \|(RT_d)^{n-1}u_{n-1} - u_{n-1}\| + \|u_{n-1} - u_n\|) \\ &\leq \|u_n - T_d(RT_d)^{n-1}u_n\| + \frac{L_1}{d}(\frac{L_1}{d}+1)\|u_n - u_{n-1}\| + \frac{L_1}{d}\|(RT_d)^{n-1}u_{n-1} - u_{n-1}\|\end{aligned} \tag{48}$$

with (38) and (45) we obtain

$$\lim_{n\to\infty}\|u_n - T_d u_n\| = 0. \tag{49}$$

Since $T_d = (1-d)I^* + dT$, we observe that

$$\begin{aligned}\|u_n - T_d u_n\| &= \|u_n - ((1-d)u_n + dTu_n)\| \\ &= \|du_n - dTu_n\| \\ &= d\|u_n - Tu_n\| = \frac{1}{\delta+1}\|u_n - Tu_n\|\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty}\|u_n - Tu_n\| = \frac{1}{d}\lim_{n\to\infty}\|u_n - T_d u_n\| = 0. \tag{50}$$

Analogously, one has

$$\|u_n - I_d u_n\| \leq \|u_n - I_d^n u_n\| + \frac{L_2}{d}(\frac{L_2}{d}+1)\|u_n - u_{n-1}\| + \frac{L_2}{d}\|I_d^{n-1}u_{n-1} - u_{n-1}\|, \tag{51}$$

which with (38) and (45) implies

$$\lim_{n\to\infty}\|u_n - I_d u_n\| = 0. \tag{52}$$

Also, since $I_d = (1-d)I^* + dI$, we observe that

$$\begin{aligned}\|u_n - I_d u_n\| &= \|u_n - ((1-d)u_n + dIu_n)\| \\ &= \|du_n - dIu_n\| \\ &= d\|u_n - Iu_n\| = \frac{1}{\delta+1}\|u_n - Iu_n\|\end{aligned}$$

Similarly, we have

$$\lim_{n\to\infty}\|u_n - Iu_n\| = \frac{1}{d}\lim_{n\to\infty}\|u_n - I_d u_n\| = 0. \tag{53}$$

□

With the above preparations, we now formulate a crucial aspect of our main results, which concerns the weak convergence of the sequence $\{u_n\}$.

Theorem 3.4. *Let E be a real uniformly convex Banach space satisfying Opial condition and let C be a nonempty closed convex subset of E. Let $I_E : E \to E$ be an identity mapping, let $T : C \to E$ with retraction R be a uniformly L_1-Lipschitzian δ–enriched asymptotically quasi-I-nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$, and let $I : C \to C$ be a uniformly L_2-Lipschitzian δ–enriched asymptotically quasi-nonexpansive mapping with a sequence $\{\eta_n\} \subset [1, \infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $\sigma_* = \inf_n \alpha_n$, $\sigma^* = \sup_n \alpha_n$, $\zeta = \sup_n k_n \geq 1$, $B = \sup_n \eta_n \geq 1$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$ and $d \in (0, 1]$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} (\frac{k_n}{d}\eta_n - 1)\alpha_n < \infty$,

(ii) $0 < \sigma_* \leq \sigma^* < \frac{1}{\frac{\zeta^2}{d^2}B^2}$,

(iii) $0 < \beta_* = \inf_n \beta_n \leq \sup_n \beta_n = \beta^* < 1$.

If the mappings $I_E - T$ and $I_E - I$ are demiclosed at zero, then the implicit iterative sequence $\{u_n\}$ defined by (8) converges weakly to a common fixed point of T and I.

Proof. Let $p \in \mathcal{F}$, then according to Lemma (3.1) the sequence $\{\|u_n - p\|\}$ converges. This provides that $\{u_n\}$ is a bounded sequence. Since E is uniformly convex, every bounded subset of E is weakly compact. Since $\{u_n\}$ is a bounded sequence in C, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $q \in C$. Hence from (50) and (53) it follows that

$$\lim_{n_k \to \infty} \|u_{n_k} - Tu_{n_k}\| = 0, \qquad \lim_{n_k \to \infty} \|u_{n_k} - Iu_{n_k}\| = 0. \tag{54}$$

Since the mappings $I_E - T$ and $I_E - I$ are demiclosed at zero, therefore, we find $Tq = q$ and $Iq = q$, which means $q \in \mathcal{F} = F(T) \cap F(I)$.

Finally, let us prove that $\{u_n\}$ converges weakly to q. In fact, suppose the contrary, that is, there exists some subsequence $\{u_{n_j}\} \subset \{u_n\}$ such that $\{u_{n_j}\}$ converges weakly to $q_1 \in C$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in \mathcal{F} = F(T) \cap F(I)$.

Taking $p = q$ and $p = q_1$ and using the same argument given in the proof of (22), we can prove that the limits $\lim_{n\to\infty} \|u_n - q\|$ and $\lim_{n\to\infty} \|u_n - q_1\|$ exist, and we have

$$\lim_{n \to \infty} \|u_n - q\| = r, \qquad \lim_{n \to \infty} \|u_n - q_1\| = r_1, \tag{55}$$

where r and r_1 are two nonnegative numbers. By virtue of the Opial condition of E, one finds

$$\begin{aligned} r &= \limsup_{n_k\to\infty} \|u_{n_k} - q\| \\ &< \limsup_{n_k\to\infty} \|u_{n_k} - q_1\| \\ &= r_1 \\ &= \limsup_{n_j\to\infty} \|u_{n_j} - q_1\| \\ &< \limsup_{n_j\to\infty} \|u_{n_j} - q\| \\ &= r. \end{aligned} \tag{56}$$

This is a contradiction. Hence $q_1 = q$. This implies that $\{u_n\}$ converges weakly to q. This completes the proof of Theorem (3.4). □

We now establish the next result regarding the strong convergence of the sequence $\{u_n\}$.

Theorem 3.5. *Let E be a real uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to E$ with retraction R be a uniformly L_1-Lipschitzian δ–enriched asymptotically quasi-I–nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1,\infty)$ and let $I : C \to C$ be a uniformly L_2-Lipschitzian δ–enriched asymptotically quasi–nonexpansive mapping with a sequence $\{\eta_n\} \subset [1,\infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $\sigma_* = \inf_n \alpha_n$, $\sigma^* = \sup_n \alpha_n$, $\zeta = \sup_n k_n \geq 1$, $B = \sup_n \eta_n \geq 1$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ and $d \in (0,1]$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} (\frac{k_n}{d}\eta_n - 1)\alpha_n < \infty$,

(ii) $0 < \sigma_* \leq \sigma^* < \frac{1}{\frac{\zeta^2}{d^2}B^2}$,

(iii) $0 < \beta_* = \inf_n \beta_n \leq \sup_n \beta_n = \beta^* < 1$.

If at least one mapping of the mappings T and I is semicompact, then the implicit iterative sequence $\{u_n\}$ defined by (8) converges strongly to a common fixed point of T and I.

Proof. Without any loss of generality, we may assume that T is semicompact. This with (49) means that there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \to u^*$ strongly and $u^* \in C$.
Since T and I are continuous, then from (50) and (53) we find

$$\|u^* - Tu^*\| = \lim_{n_k\to\infty} \|u_{n_k} - Tu_{n_k}\| = 0, \qquad \|u^* - Iu^*\| = \lim_{n_k\to\infty} \|u_{n_k} - Iu_{n_k}\| = 0. \tag{57}$$

This shows that $u^* \in \mathcal{F} = F(T) \cap F(I)$. According to Lemma (3.1) the limit $\lim_{n\to\infty} \|u_n - u^*\|$ exists. Then

$$\lim_{n\to\infty} \|u_n - u^*\| = \lim_{n_k\to\infty} \|u_{n_k} - u^*\| = 0, \tag{58}$$

which means that $\{u_n\}$ converges to $u^* \in \mathcal{F}$. This completes the proof of Theorem (3.5). □

Corollary 3.6. *Under the conditions of Lemma 3.1, the convergence factor in estimate (19) is* $1 + \phi_n$, *where*

$$\phi_n = \frac{(\frac{k_n^2}{d^2}\eta_n^2 - 1)\alpha_n}{1 - \sigma^* \frac{\zeta^2}{d^2} B^2}.$$

The asymptotic convergence behavior is governed by $\limsup_{n\to\infty}(1+\phi_n)^{1/n}$. *Since* $d = \frac{1}{\delta+1}$, *the enrichment parameter* $\delta \in [0, \infty)$ *determines the contraction rate.*

From condition (ii) of Lemma 3.1, the admissible values of δ *satisfy*

$$0 \le \delta < \frac{1}{\sqrt{\sigma^* \zeta B}} - 1,$$

provided the right-hand side is positive. The boundary value

$$\delta_{opt} := \frac{1}{\sqrt{\sigma^* \zeta B}} - 1$$

(when nonnegative) gives the largest admissible enrichment parameter, which minimizes the asymptotic contraction factor and therefore yields the fastest guaranteed convergence.

If $\frac{1}{\sqrt{\sigma^* \zeta B}} - 1 < 0$, *then the admissible range is empty for* $\delta > 0$; *in this case the optimal choice is* $\delta_{opt} = 0$, *which recovers the standard (non-enriched) implicit iteration scheme studied by Mukhamedov and Saburov (2010).*

Theorem 3.7. *Let* $\{u_n\}$ *be the ideal sequence defined by (8) and suppose it converges strongly to* $p \in \mathcal{F}$ *(e.g., under the conditions of Theorem 3.5). Let* $\{\epsilon_n\}$ *be a sequence of nonnegative real numbers with* $\sum_{n=1}^{\infty} \epsilon_n < \infty$, *and let* $\{u_n^\epsilon\}$ *be a perturbed sequence satisfying*

$$\|u_n^\epsilon - ((1-\alpha_n)u_{n-1}^\epsilon + \alpha_n T_d (RT_d)^{n-1} v_n^\epsilon)\| \le \epsilon_n$$

for all $n \ge 1$. *Assume additionally that:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$;

(ii) The Lipschitz constants satisfy $\sigma^* \frac{L^2}{d^2} < 1$, *where* $L := \max\{1, L_1, L_2\}$, $\sigma^* = \sup_n \alpha_n$ *and* $d \in (0, 1]$.

Then $\{u_n^\epsilon\}$ *converges strongly to* p.

Proof. Define the error sequence $e_n = u_n^\epsilon - u_n$ and let $L = \max\{1, L_1, L_2\}$.

From the definitions of the ideal and perturbed sequences, we have:

$$u_n = (1-\alpha_n)u_{n-1} + \alpha_n T_d (RT_d)^{n-1} v_n, \tag{59}$$

$$u_n^\epsilon = (1-\alpha_n)u_{n-1}^\epsilon + \alpha_n T_d (RT_d)^{n-1} v_n^\epsilon + \theta_n, \tag{60}$$

where $\|\theta_n\| \leq \epsilon_n$.

Subtracting (59) from (60) we obtain

$$e_n = (1-\alpha_n)e_{n-1} + \alpha_n(T_d(RT_d)^{n-1}v_n^\epsilon - T_d(RT_d)^{n-1}v_n) + \theta_n. \tag{61}$$

Using the Lipschitz properties of $T_d(RT_d)^{n-1}$ and I_d^n with constant L, we obtain

$$\|T_d(RT_d)^{n-1}v_n^\epsilon - T_d(RT_d)^{n-1}v_n\| \leq \frac{L}{d}\|v_n^\epsilon - v_n\|, \tag{62}$$

$$\|v_n^\epsilon - v_n\| \leq \frac{L}{d}\|e_n\|. \tag{63}$$

Substituting (63) into (62) yields

$$\|T_d(RT_d)^{n-1}v_n^\epsilon - T_d(RT_d)^{n-1}v_n\| \leq \frac{L^2}{d^2}\|e_n\|. \tag{64}$$

Taking norms in (61) and using (64)

$$\|e_n\| \leq (1-\alpha_n)\|e_{n-1}\| + \alpha_n\frac{L^2}{d^2}\|e_n\| + \epsilon_n. \tag{65}$$

Since $\sigma^*\frac{L^2}{d^2} < 1$, we have $1-\alpha_n\frac{L^2}{d^2} \geq 1-\sigma^*\frac{L^2}{d^2} > 0$. Rearranging, we get

$$(1-\alpha_n\frac{L^2}{d^2})\|e_n\| \leq (1-\alpha_n)\|e_{n-1}\| + \epsilon_n. \tag{66}$$

Hence

$$\|e_n\| \leq \frac{1-\alpha_n}{1-\alpha_n\frac{L^2}{d^2}}\|e_{n-1}\| + \frac{\epsilon_n}{1-\sigma^*\frac{L^2}{d^2}}. \tag{67}$$

Observe that

$$\frac{1-\alpha_n}{1-\alpha_n\frac{L^2}{d^2}} = 1 + \frac{\alpha_n(\frac{L^2}{d^2}-1)}{1-\alpha_n\frac{L^2}{d^2}} \leq 1 + \frac{\frac{L^2}{d^2}-1}{1-\sigma^*\frac{L^2}{d^2}}\alpha_n.$$

Let $C = \frac{\frac{L^2}{d^2}-1}{1-\sigma^*\frac{L^2}{d^2}}$. Then

$$\|e_n\| \leq (1+C\alpha_n)\|e_{n-1}\| + \frac{\epsilon_n}{1-\sigma^*\frac{L^2}{d^2}}. \tag{68}$$

Since $\sum_{n=1}^{\infty}\alpha_n < \infty$ and $\sum_{n=1}^{\infty}\epsilon_n < \infty$, Lemma 2.3 implies that $\lim_{n\to\infty}\|e_n\|$ exists. Let this limit be ℓ. We now show $\ell = 0$. From (66) we have

$$\|e_n\| \leq \frac{1-\alpha_n}{1-\alpha_nL^2/d^2}\|e_{n-1}\| + \frac{\epsilon_n}{1-\sigma^*L^2/d^2}.$$

Observe that $\frac{1-\alpha_n}{1-\alpha_nL^2/d^2} = 1 + \frac{\alpha_n(L^2/d^2-1)}{1-\alpha_nL^2/d^2}$.

Since $\sum \alpha_n < \infty$ and $L^2/d^2 - 1 \geq 0$, the sequence $\{\frac{1-\alpha_n}{1-\alpha_n L^2/d^2}\}$ satisfies

$$\sum_{n=1}^{\infty} \left(\frac{1-\alpha_n}{1-\alpha_n L^2/d^2} - 1 \right) < \infty.$$

Moreover, $\sum \frac{\epsilon_n}{1-\sigma^* L^2/d^2} < \infty$. By Lemma 2 of Liu (2001), the limit $\ell = 0$.

Therefore $u_n^{\epsilon} \to p$ strongly, completing the proof of Theorem 3.7. □

Conclusion

In this paper, we introduce and analyze a novel implicit iterative process for approximating common fixed points of a pair of nonlinear mappings in real Banach spaces. Specifically, we consider a δ-enriched asymptotically quasi-I-nonexpansive nonself mapping $T : C \to E$ and a δ-enriched asymptotically quasi-nonexpansive mapping $I : C \to C$. The proposed scheme, defined in (8), serves as a substantial generalization of classical implicit iteration methods.

The main contributions of this work are summarized as follows.

- Lemma 3.1 establishes the existence of $\lim_{n\to\infty} \|u_n - p\|$ for each common fixed point p under the conditions $\sum_{n=1}^{\infty} (\frac{k_n}{d}\eta_n - 1)\alpha_n < \infty$ and $\sigma^* < \frac{1}{\frac{\zeta^2}{d^2}B^2}$.
- Theorem 3.2 provides a necessary and sufficient condition for strong convergence, namely $\liminf_{n\to\infty} d(u_n, \mathcal{F}) = 0$.
- Proposition 3.3 shows that in uniformly convex Banach spaces, the sequence satisfies $\lim_{n\to\infty} \|u_n - Tu_n\| = 0 = \lim_{n\to\infty} \|u_n - Iu_n\|$.
- Theorem 3.4 establishes weak convergence to a common fixed point in uniformly convex Banach spaces satisfying Opial's condition, assuming the demiclosedness of $I_E - T$ and $I_E - I$ at zero.
- Theorem 3.5 proves strong convergence under the semicompactness of at least one of the mappings.
- An explicit formula for the optimal enrichment parameter is derived: $\delta_{\text{opt}} = \frac{1}{\sqrt{\sigma^* \zeta B}} - 1$, which minimizes the convergence factor. When the right-hand side is negative, the standard (non-enriched) scheme is recovered.
- Theorem 3.7 provides a stability analysis, demonstrating that perturbed sequences converge to the same fixed point whenever $\sum \alpha_n < \infty$ and $\sigma^* \frac{L^2}{d^2} < 1$.

Collectively, these results unify, generalize, and extend several important convergence theorems in the literature, including those of Xu and Ori (2001), Sun (2003), Gu and Lu (2006), and Mukhamedov and Saburov (2010). The introduction of the enrichment parameter δ and the associated averaged operators T_d^n and I_d^n yields a flexible framework that accommodates a broader class of nonlinear mappings while retaining desirable convergence properties.

Declaration

Availability of Data and Material

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Competing Interests

The authors declare that there is no conflict of interest.

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